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Recurrence Relations for Single Moments of Generalized Order Statistics from Doubly Truncated Continuous Distributions

Bakheet N. Al-Matrafi and **Taghreed M. Jawa**

Department of Mathematics & Statistics, Taif University, Saudi Arabia
Department of Mathematics, Girls Education Colleges, Makkah, Saudi Arabia

*Abstract***.** In this paper, we consider a general class of continuous distributions given by $1-F(x)=\left[ah(x)+b\right]^{c}$. A recurrence relation for single higher moments of generalized order statistics from the doubly truncated case of the above class is derived. Recurrence relations for single higher moments of ordinary order statistics and *k*-records (ordinary record values when $k=1$), have been obtained as special cases from generalized order statistics. These results are utilized to establish similar recurrence relations for Weibull, Pareto and Power function distributions.

*Keywords***:** Record values; single higher moments.

1. Introduction

Recurrence relations for moments of generalized order statistics (*gos's*) have been discussed by [1-8], among others. In [9] some relations for moments of *gos's* of a general class of continuous distributions have obtained. In this paper we obtain a recurrence relation for single higher moments of *gos's* in the case of doubly truncated distributions. Suppose that the random variable X has a distribution function (*df*) of the following class of absolutely continuous distribution:

$$F(x)=1-\left[ah(x)+b\right]^{c}, \quad a\neq 0, \quad c\neq 0, \quad \alpha\leq x\leq\beta, \tag{1.1}$$

and probability density function (*pdf*)

$$f(x) = -ach'(x)\left[ah(x)+b\right]^{c-1}. \quad (1.2)$$

It is well known that the doubly truncated *pdf*, say $f_d(x)$, is defined as

$$f_d(x) = \frac{f(x)}{Q-P}, \quad \alpha \le P_1 \le x \le Q_1 \le \beta\,,$$

where $P = F(P_1)$ and $Q = F(Q_1)$.

The *df* and *pdf* of the doubly truncated case of (1.1) and (1.2) are given by

$$F_d(x) = \frac{F(x)-P}{Q-P}, \quad (1.3)$$

$$f_d(x) = \left(\frac{\overline{F}(x)}{Q-P}\right)\left(\frac{-ach'(x)}{ah(x)+b}\right), \quad \alpha \le P_1 \le x \le Q_1 \le \beta\,. \quad (1.4)$$

2. Recurrence Relations for Single Moments of *GOS'S*

Let $X_{1;n,m,k}, X_{2;n,m,k}, \ldots, X_{n;n,m,k}$ be n *gos's* from the *pdf* (1.2), ($n > 1$, m and k are real numbers and $k \ge 1$).

The *pdf* of $X_{r;n,m,k}$ is given by [1] as follows

$$f_{X_{r;n,m,k}}(x) = \frac{C_{r-1}}{(r-1)!} g_m^{r-1}(F(x))[\overline{F}(x)]^{\gamma_r - 1} f(x), \quad x \in \chi, \quad (2.1)$$

where χ is the domain on which $f_{X_{r;n,m,k}}(x)$ is positive and

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i\,, \quad \gamma_i = k + (n-i)(m+1)\,,$$

and, for $0 < z < 1$

$$g_m(z) = \begin{cases} [1-(1-z)^{m+1}]\,/(m+1)\,, & m \ne -1 \\ -ln(1-z)\,, & m = -1 \end{cases}.$$

The j^{th} moment of the r^{th} *gos* can be obtained, for $j \geq 1$, from (2.1), as

$$\mu_{r;n,m,k}^{(j)} \equiv E[X_{r;n,m,k}^{j}] = \frac{C_{r-1}}{(r-1)!}\int_{\chi} x^j g_m^{r-1}(F(x))[\overline{F}(x)]^{\gamma_r - 1} f(x)\,dx\,. \quad (2.2)$$

The j^{th} moment of ordinary order statistics (*oos's*) and *k*-records can be obtained, from (2.2) by putting $m = 0$, $k = 1$ and $m = -1$, $k \geq 1$, respectively.

A recurrence relation for higher moments of *gos's* can be obtained in the following theorem.

Theorem 2.1

Let *X* be a *r.v.* with $F_d(x)$ defined on (P_1, Q_1) by (1.3) then for real numbers m, k with $m \geq -1$, $k \geq 1$ and integers $r, j \geq 1$, the recurrence relation

$$\mu_{r;n,m,k}^{(j)} - \mu_{r-1;n,m,k}^{(j)} = \begin{cases} \frac{j}{ac\gamma_r}\Big((1-Q)E\big[W(X_{r;n,m,k})\big] - E\big[G(X_{r;n,m,k})\big]\Big), & m \neq -1 \\ \frac{j}{ack}\Big((1-Q)E\big[W(X_{r;n,m,k})\big] - E\big[G(X_{r;n,m,k})\big]\Big), & m = -1 \end{cases}, \quad (2.3)$$

is satisfied, where $G(x) = \frac{x^{j-1}}{h'(x)}\big(ah(x)+b\big)$ and

$W(x) = \big(ah(x)+b\big)^{-c}\, G(x)$.

Proof

From (2.2) we can write

$$\mu_{r;n,m,k}^{(j)} = \frac{C_{r-1}}{\gamma_r (r-1)!}\int_{P_1}^{Q_1} x^j g_m^{r-1}(F_d(x))\, d[-(\overline{F}_d(x))]^{\gamma_r}.$$

Integrating by parts, we obtain

$$\mu_{r;n,m,k}^{(j)} = \frac{(r-1)C_{r-1}}{(r-1)!}\int_{P_1}^{Q_1} x^j g_m^{r-2}(F_d(x))[\overline{F}_d(x)]^{\gamma_r + m} f_d(x)\,dx$$
$$+ \frac{jC_{r-1}}{\gamma_r (r-1)!}\int_{P_1}^{Q_1} x^{j-1} g_m^{r-1}(F_d(x))[\overline{F}_d(x)]^{\gamma_r}\,dx,$$

which can be written as

$$\mu^{(j)}_{r;n,m,k} = \frac{C_{r-2}}{(r-2)\,!}\int_{P_1}^{Q_1} x^j g_m^{r-2}(F_d(x))[\overline{F}_d(x)]^{\gamma_{r-1}-1} f_d(x)\,dx$$
$$+\frac{jC_{r-1}}{\gamma_r(r-1)\,!}\int_{P_1}^{Q_1} x^{j-1} g_m^{r-1}(F_d(x))[\overline{F}_d(x)]^{\gamma_r}\,dx,$$

or equivalently,

$$\mu^{(j)}_{r;n,m,k} - \mu^{(j)}_{r-1;n,m,k} = \frac{jC_{r-1}}{\gamma_r(r-1)\,!}\int_{P_1}^{Q_1} x^{j-1} g_m^{r-1}(F_d(x))[\overline{F}_d(x)]^{\gamma_r}\,dx, \tag{2.4}$$

since, $C_{r-1} = \gamma_r C_{r-2}$, $\gamma_r + m = k + (n-r)(m+1) + m = \gamma_{r-1} - 1$.

Eq.(2.4) can be rewritten as

$$\mu^{(j)}_{r;n,m,k} - \mu^{(j)}_{r-1;n,m,k} = \frac{jC_{r-1}}{\gamma_r(r-1)\,!}\int_{P_1}^{Q_1} x^{j-1} g_m^{r-1}(F_d(x))[\overline{F}_d(x)]^{\gamma_r-1}\,[\overline{F}_d(x)]dx, \tag{2.5}$$

making use of $\overline{F}_d(x) = \frac{1-Q}{Q-P} + \frac{ah(x)+b}{ach'(x)} f_d(x)$ in (2.5), we obtain,

$$\mu^{(j)}_{r;n,m,k} - \mu^{(j)}_{r-1;n,m,k} = \frac{-Q_2 jC_{r-1}}{\gamma_r\Gamma(r)}\int_{P_1}^{Q_1} x^{j-1} g_m^{r-1}(F_d(x))[\overline{F}_d(x)]^{\gamma_r-1}\,dx$$
$$-\frac{jC_{r-1}}{\gamma_r\Gamma(r)}\int_{P_1}^{Q_1} x^{j-1} g_m^{r-1}(F_d(x))[\overline{F}_d(x)]^{\gamma_r-1}\left(\frac{ah(x)+b}{ach'(x)}\right) f_d(x)dx, \tag{2.6}$$

where $Q_2 = \frac{1-Q}{Q-P}$. Since, $\frac{(Q-P)f_d(x)}{[ah(x)+b]^{c-1}[-ach'(x)]} = 1$, we can rewrite (2.6) as

$$\mu^{(j)}_{r;n,m,k} - \mu^{(j)}_{r-1;n,m,k} = \frac{-Q_2 jC_{r-1}}{\gamma_r\Gamma(r)}\int_{P_1}^{Q_1} x^{j-1} g_m^{r-1}(F_d(x))[\overline{F}_d(x)]^{\gamma_r-1}$$
$$\times\frac{(Q-P)}{[ah(x)+b]^{c-1}}\frac{f_d(x)}{-ach'(x)}\,dx$$
$$-\frac{jC_{r-1}}{\gamma_r\Gamma(r)}\int_{P_1}^{Q_1} x^{j-1} g_m^{r-1}(F_d(x))[\overline{F}_d(x)]^{\gamma_r-1}$$
$$\times\left(\frac{ah(x)+b}{ach'(x)}\right) f_d(x)dx\ . \tag{2.7}$$

Let $G(x)=\dfrac{x^{j-1}}{h'(x)}\big(ah(x)+b\big)$, $W(x)=G(x)\big(ah(x)+b\big)^{-c}$. Thus,

$$\mu^{(j)}_{r;n,m,k}-\mu^{(j)}_{r-1;n,m,k}=\frac{j(1-Q)C_{r-1}}{ac\gamma_r\Gamma(r)}\int_{P_1}^{Q_1}W(x)g_m^{r-1}(F_d(x))[\overline{F}_d(x)]^{\gamma_r-1}f_d(x)dx$$
$$-\frac{jC_{r-1}}{ac\gamma_r\Gamma(r)}\int_{P_1}^{Q_1}G(x)g_m^{r-1}(F_d(x))[\overline{F}_d(x)]^{\gamma_r-1}f_d(x)dx,$$

or equivalently,

$$\mu^{(j)}_{r;n,m,k}-\mu^{(j)}_{r-1;n,m,k}=\begin{cases}\dfrac{j}{ac\gamma_r}\Big((1-Q)E\big[W(X_{r;n,m,k})\big]-E\big[G(X_{r;n,m,k})\big]\Big), & m\neq-1\\ \dfrac{j}{ack}\Big((1-Q)E\big[W(X_{r;n,m,k})\big]-E\big[G(X_{r;n,m,k})\big]\Big), & m=-1\end{cases},$$

as desired.

Remarks 2.2. **One can note the following special cases**

(1) The doubly truncated case of a distribution is the most general case since it includes the right truncated, left truncated and non-truncated distributions as special cases.

(2) In the left truncated $(Q=1)$, Eq.(2.3) reduces to

$$\mu^{(j)}_{r;n,m,k}-\mu^{(j)}_{r-1;n,m,k}=\begin{cases}\dfrac{-j}{ac\gamma_r}E\big[G(X_{r;n,m,k})\big], & m\neq-1\\ \dfrac{-j}{ack}E\big[G(X_{r;n,m,k})\big], & m=-1\end{cases}. \tag{2.8}$$

(3) In the non-truncated case $(Q=1,P=0)$, Eq.(2.3) reduces to relation (2.8), which means that the relations in the left truncated and non-truncated cases are similar in the form in spite of the difference in their domains. Note that Eq.(2.8) coincides with the result obtained by [9].

(4) In the right truncated case $(P=0)$ the recurrence relation is given by Eq.(2.3).

(5) In the case of *oos's* $(m=0,\quad k=1\ and\ \gamma_r=n-r+1)$, Eqs.(2.3) and (2.8) are reduced, respectively, to

$$\left.\begin{aligned}\mu^{(j)}_{r:n}-\mu^{(j)}_{r-1:n}&=\frac{j}{ac(n-r+1)}\big((1-Q)E[W(X_{r,n})]-E[G(X_{r,n})]\big)\\ \mu^{(j)}_{r:n}-\mu^{(j)}_{r-1:n}&=\frac{-j}{ac(n-r+1)}E[G(X_{r,n})]\end{aligned}\right\}. \tag{2.9}$$

(6) In the case of *k*-records $(m=-1,\quad k\geq 1)$, Eqs.(2.3) and (2.8), respectively become,

$$\left.\begin{aligned}\mu^{(j)}_{U_k(r)}-\mu^{(j)}_{U_k(r-1)}&=\frac{j}{ack}\big((1-Q)E[W(X_{U_k(r)})]-E[G(X_{U_k(r)})]\big)\\ \mu^{(j)}_{U_k(r)}-\mu^{(j)}_{U_k(r-1)}&=\frac{-j}{ack}E[G(X_{U_k(r)})]\end{aligned}\right\}. \tag{2.10}$$

Note that for $k=1,$ we obtain the *orv's* case.

3. Special Cases

In this section, three members of class (1.3) are used to illustrate the derived relations in these cases. These members are Weibull, Pareto and Power function distributions.

(1) Weibull Distribution

Choosing $a=1, b=0, c=1, h(x)=e^{-\theta x^p}$, in (1.1) one has

$\overline{F}(x)=e^{-\theta x^p}$, $x\geq 0$,

then, $G(x)=\dfrac{-x^{j-p}}{p\theta}$, $W(x)=\dfrac{-x^{j-p}}{p\theta e^{-\theta x^p}}$.

The relation (2.8), reduces to

$$\mu^{(j)}_{r;n,m,k} - \mu^{(j)}_{r-1;n,m,k} = \begin{cases} \dfrac{j}{p\theta\gamma_r}\left(\mu^{(j-p)}_{r;n,m,k} - (1-Q)E\left[X^{j-p}_{r;n,m,k}\, e^{\theta X^{p}_{r;n,m,k}}\right]\right), m \neq -1 \\ \dfrac{j}{p\theta k}\left(\mu^{(j-p)}_{r;n,m,k} - (1-Q)E\left[X^{j-p}_{r;n,m,k}\, e^{\theta X^{p}_{r;n,m,k}}\right]\right), m = -1 \end{cases}.$$

The relation (2.8), reduces to

$$\mu^{(j)}_{r;n,m,k} - \mu^{(j)}_{r-1;n,m,k} = \begin{cases} \dfrac{j}{p\theta\gamma_r}\mu^{(j-p)}_{r;n,m,k}, \quad m \neq -1 \\ \dfrac{j}{p\theta k}\mu^{(j-p)}_{r;n,m,k}, \quad m = -1 \end{cases}.$$

The relation (2.9), reduces to

$$\mu^{(j)}_{r:n} - \mu^{(j)}_{r-1:n} = \frac{j}{p\theta(n-r+1)}\left(\mu^{(j-p)}_{r:n} - (1-Q)E\left[X^{j-p}_{r:n}\, e^{\theta X^{p}_{r:n}}\right]\right),$$

$$\mu^{(j)}_{r:n} - \mu^{(j)}_{r-1:n} = \frac{j}{p\theta(n-r+1)}\mu^{(j-p)}_{r:n} \quad .$$

The relation (2.10), reduces to

$$\mu^{(j)}_{U_k(r)} - \mu^{(j)}_{U_k(r-1)} = \frac{j}{p\theta k}\left(\mu^{(j-p)}_{U_k(r)} - (1-Q)E\left[X^{j-p}_{U_k(r)}\, e^{\theta X^{p}_{U_k(r)}}\right]\right),$$

$$\mu^{(j)}_{U_k(r)} - \mu^{(j)}_{U_k(r-1)} = \frac{j}{p\theta k}\mu^{(j-p)}_{U_k(r)} .$$

The *orv's* case is obtained for $k = 1$.

The exponential and Rayleigh distributions are obtained for the choice $p = 1$, and $p = 2$, respectively.

(2) Pareto Distribution

Choosing $a = 1, b = 0, c = 1, h(x) = x^{-p}$, in (1.1) one has $\overline{F}(x) = x^{-p}, x \geq 1$ which is the Pareto *d.f.* Also one has $G(x) = \dfrac{-x^{j}}{p}$, $W(x) = \dfrac{-x^{j+p}}{p}$.

The relation (2.8), reduces to

$$\mu^{(j)}_{r;n,m,k} - \mu^{(j)}_{r-1;n,m,k} = \begin{cases} \dfrac{j}{p\gamma_r}\left(\mu^{(j)}_{r;n,m,k} - (1-Q)\mu^{(j+p)}_{r;n,m,k}\right), & m \neq -1 \\ \dfrac{j}{pk}\left(\mu^{(j)}_{r;n,m,k} - (1-Q)\mu^{(j+p)}_{r;n,m,k}\right), & m = -1 \end{cases}.$$

The relation (2.8), reduces to

$$\mu^{(j)}_{r;n,m,k} - \mu^{(j)}_{r-1;n,m,k} = \begin{cases} \dfrac{j}{p\gamma_r}\mu^{(j)}_{r;n,m,k}, & m \neq -1 \\ \dfrac{j}{pk}\mu^{(j)}_{r;n,m,k}, & m = -1 \end{cases}.$$

The relation (2.9), reduces to

$$\mu^{(j)}_{r:n} - \mu^{(j)}_{r-1:n} = \frac{j}{p(n-r+1)}\left(\mu^{(j)}_{r:n} - (1-Q)\mu^{(j+p)}_{r:n}\right),$$

$$\mu^{(j)}_{r:n} - \mu^{(j)}_{r-1:n} = \frac{j}{p(n-r+1)}\mu^{(j)}_{r:n}.$$

The relation (2.10), reduces to

$$\mu^{(j)}_{U_k(r)} - \mu^{(j)}_{U_k(r-1)} = \frac{j}{pk}\left(\mu^{(j)}_{U_k(r)} - (1-Q)\mu^{(j+p)}_{U_k(r)}\right),$$

$$\mu^{(j)}_{U_k(r)} - \mu^{(j)}_{U_k(r-1)} = \frac{j}{pk}\mu^{(j)}_{U_k(r)}.$$

The $orv's$ case is obtained for $k=1$.

(3) Power Function Distribution

Choosing $a=-1, b=1, c=1, h(x)=x^p$, in (1.1) gives that $\overline{F}(x) = 1-x^p, 0 \le x \le 1$. Hence $G(x) = \dfrac{x^{j-p} - x^j}{p}$, $W(x) = \dfrac{x^{j-p}}{p}$.

The relation (2.3), reduces to

$$\mu^{(j)}_{r;n,m,k} - \mu^{(j)}_{r-1;n,m,k} = \begin{cases} \frac{-j}{p\gamma_r}\left(\mu^{(j)}_{r;n,m,k} - Q\mu^{(j-p)}_{r;n,m,k}\right), & m \neq -1 \\ \frac{-j}{pk}\left(\mu^{(j)}_{r;n,m,k} - Q\mu^{(j-p)}_{r;n,m,k}\right), & m = -1 \end{cases}.$$

The relation (2.8), reduces to

$$\mu^{(j)}_{r;n,m,k} - \mu^{(j)}_{r-1;n,m,k} = \begin{cases} \frac{-j}{p\gamma_r}\mu^{(j)}_{r;n,m,k} \quad , & m \neq -1 \\ \frac{-j}{pk}\mu^{(j)}_{r;n,m,k} \quad , & m \neq -1 \end{cases}.$$

The relation (2.9), reduces to

$$\mu^{(j)}_{r:n} - \mu^{(j)}_{r-1:n} = \frac{-j}{p(n-r+1)}\left(\mu^{(j)}_{r:n} - Q\mu^{(j-p)}_{r:n}\right),$$

$$\mu^{(j)}_{r:n} - \mu^{(j)}_{r-1:n} = \frac{-j}{p(n-r+1)}\mu^{(j)}_{r:n}.$$

The relation (2.10), reduces to

$$\mu^{(j)}_{U_k(r)} - \mu^{(j)}_{U_k(r-1)} = \frac{-j}{pk}\left(\mu^{(j)}_{U_k(r)} - Q\mu^{(j-p)}_{U_k(r)}\right),$$

$$\mu^{(j)}_{U_k(r)} - \mu^{(j)}_{U_k(r-1)} = \frac{-j}{pk}\mu^{(j)}_{U_k(r)}.$$

Note that the *orv's* case is obtained for $k = 1$.

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علاقات تكرارية للعزوم المفردة للإحصاءات المرتبة المعممة لتوزيعات متصلة مقطوعة الطرفين

بخيت المطرفي – و تغريد جاوا

قسم الرياضيات والإحصاء، جامعة الطائف، والمملكة العربية السعودية

قسم الرياضيات، كلية التربية للبنات، مكة المكرمة، المملكة العربية السعودية

المستخلص. في هذا البحث تم دراسة عائلة عامة من التوزيعات المتصلة والتي على الصورة : $1-F(x)=\left[ah(x)+b\right]^{c}$ حيث تم استنتاج بعض العلاقات التكرارية للعزوم المفردة للإحصاءات المرتبة المعممة لعائلة التوزيع السابقة المقطوعة الطرفين.

كذلك فإن العزوم المفردة من الرتب العليا للإحصاءات المرتبة العادية والقيم المسجلة من الرتبة k (القيم المسجلة العدية عندما $k=1$) قد تم الحصول عليها كحالات خاصة. وهذه النتائج يمكن استخدامها في الحصول على نتائج مماثلة لتوزيعات وايبل، باريتو، ودالة القوى كحالات خاصة من هذه العائلة.